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NUMERICAL SOLUTION OF THE NAVIER-STOKES EQUATIONS BY MEANS OF A MULTIGRID METHOD AND NEWTON-ITERATION

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Numerical solution of the Navier-Stokes equations by means of a multigrid method and Newton-iteration*)

by

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ABSTRACT

In this report a multigrid method for the solution of elliptic boundary value problems in a rectangle is considered. A 7-point restriction and prolongation operator is introduced, with which a Galerkin approximation can be defined as coarse grid operator. A 7-point incomplete LU-decomposition is chosen as smoothing operator. It is shown that the method is fast and robust for a large variety of problems. Especially some numerical experiments on the Navier-Stokes equations are reported: the driven cavity and the flow around a cylinder.

KEY WORDS & PHRASES: *multigrid methods, 7-point restriction and prolongation, Galerkin approximation, 7-point incomplete LU-decomposition*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

Multigrid methods have been investigated by BRANDT (1977, 1979), FREDERICKSON (1975), HACKBUSH (1978), NICOLAIDES (1979), WESSELING and SONNEVELD (1980) and WESSELING (1980).

In this report a multigrid method is described with some novel features: a 7-point prolongation and restriction and a 7-point incomplete LU-decomposition as smoothing operator.

In Section 2 we give a description of a large class of multigrid methods. Our algorithm is obtained by special choices of some parameters and the prolongation, restriction, coarse-grid and smoothing operators.

In Section 3 we give some arguments why a 7-point incomplete LU-decomposition as smoothing operator is used.

In Section 4 some numerical experiments are reported on the Navier-Stokes equations.

2. MULTIGRID METHODS

We consider a linear elliptic partial differential equation denoted by:

$$Au = f \tag{2.1}$$

and valid in the unit square $\Omega = \{(x,y) \mid 0 < x < 1,\ 0 < y < 1\}$. Boundary conditions are defined on the boundary $\partial\Omega$ of Ω. A computational grid Ω^ℓ and a corresponding set of grid-functions U^ℓ are defined by:

$$\begin{aligned} \Omega^\ell &= \{(x_i,y_j) \mid x_i = i.2^{-\ell},\ y_j = j.2^{-\ell},\ i = 0(1)2^\ell,\ j = 0(1)2^\ell\}, \\ U^\ell &= \{u^\ell : \Omega^\ell \to \mathbb{R}\}. \end{aligned} \tag{2.2}$$

After discretization of (2.1) and the boundary conditions we obtain an algebraic system of equations denoted by:

$$A^\ell u^\ell = f^\ell, \tag{2.3}$$

with $A^\ell: U^\ell \to U^\ell$.

The multigrid method makes use of a hierarchy of computational grids Ω^k and corresponding sets of grid-functions U^k, $k = \ell-1(-1)1$, defined by (2.2) with ℓ replaced by k. On the coarser grids (2.3) is approximated by:

$$(2.4) \qquad A^k u^k = f^k, \quad k = \ell-1(-1)1,$$

with A^k some suitably chosen coarse grid operator. A restriction operator R^k and a prolongation operator P^k are introduced:

$$(2.5) \qquad R^k: U^k \to U^{k-1}, \quad P^k: U^{k-1} \to U^k.$$

Finally, we define a smoothing operator on each reduction level k:

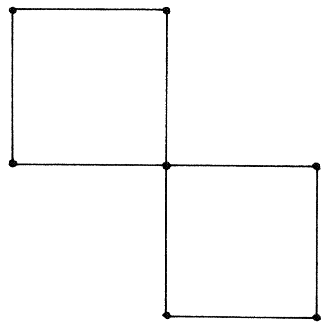
$$(2.6) \qquad u^k := S(k,A^k,u^k,f^k).$$

A class of multigrid methods can be described in quasi-Algol as follows:

```
procedure multigrid (k,u,f); integer k; array u,f;
begin     integer q;
          if k=1 then u^k:=(A^k)^-1 f^k
          else
          begin for q:=1(1)qa[k] do u^k:=S(k,A^k,u^k,f^k);
                f^(k-1):=R^k(f^k-A^k u^k); u^(k-1):=0;
                for q:=1(1)qc[k] do multigrid (k-1,u,f);
                u^k:=u^k+P^k u^(k-1);
                for q:=1(1)qb[k] do u^k:=S(k,A^k,u^k,f^k);
          end;
```

One execution of multigrid (ℓ,u,f) will be defined as one multigrid iteration.

Most multigrid strategies described in the literature can be obtained as cases of the foregoing algorithm for special choices of the parameters qa[k], qb[k], qc[k] and the operators P^k, R^k, A^k and $S(k,A^k,u^k,f^k)$. Our multigrid strategy will be described for the case that (2.3) is a 7-point



Finite difference molecule of (2.3)

finite difference approximation to a general second order elliptic partial differential equation (2.1) containing mixed derivatives. The difference molecule is given in the accompanying figure.

Furthermore, the following choices are made $(k = \ell-1(-1)2)$:

Parameters:

(2.7) $\quad qa[k] = 0, \quad qb[k] = 1, \quad qc[k] = 1.$

Restriction:

(2.8)
$$(R^k u^k)_{i,j} = \frac{1}{4} u^k_{2i,2j} + \frac{1}{8}(u^k_{2i+1,2j} + u^k_{2i-1,2j} + u^k_{2i,2j-1} + u^k_{2i+1,2j-1} + u^k_{2i-1,2j+1})$$

Prolongation:

(2.9)
$$(P^k u^{k-1})_{2i,2j} = u^{k-1}_{i,j}; \qquad (P^k u^{k-1})_{2i+1,2j} = \frac{1}{2}(u^{k-1}_{i,j} + u^{k-1}_{i+1,j})$$

$$(P^k u^{k-1})_{2i,2j+1} = \frac{1}{2}(u^{k-1}_{i,j} + u^{k-1}_{i,j+1});$$

$$(P^k u^{k-1})_{2i+1,2j+1} = \frac{1}{2}(u^{k-1}_{i+1,j} + u^{k-1}_{i,j+1})$$

Coarse grid operator:

(2.10) $\quad A^{k-1} = R^k A^k P^k.$

Smoothing operator:

(2.11) $\quad S(k, A^k, u^k, f^k) = u^k + B^k(f^k - A^k u^k),$

with B^k the 7-point incomplete LU-decomposition (ILU-7) of A^k

(2.12) $$B^k = (\tilde{L}^k\tilde{U}^k)^{-1}.$$

The matrices $\tilde{L}^k$ and $\tilde{U}^k$ are constructed as described by WESSELING and SONNEVELD (1980) with whom the use of ILU-decomposition for smoothing in the multigrid method originates.

Another novel feature in the present method is the use of 7-point restriction (2.8) and prolongation (2.9) operators. The use of Galerkin approximations for the coarse grid operators according to (2.10) has been considered by FREDERICKSON (1975). HACKBUSH (1978), WESSELING and SONNEVELD (1980) and WESSELING (1980). BRANDT (1977, 1979) takes for A^k, $k = \ell-1(-1)1$ finite difference approximations: (2.3) with ℓ replaced by k.

3. SMOOTHING ANALYSIS AND SOME NUMERICAL EXPERIMENTS

For smoothing analyses based on Fourier mode analysis, I refer to BRANDT (1977). He found for point and line Gauss Seidel applied to the usual 5-point descretization of the Poisson equation smoothing factors $\mu = 0.50$ and $\bar{\mu} = 1/\sqrt{5} \cong 0.447$ respectively. In the same way, we can find smoothing factors for 5-point and 7-point incomplete LU-decomposition. Using his notation we obtain as convergence factor $\mu(\theta)$ for the ILU-5.

(3.1) $$\mu(\theta) = \frac{a.\cos(\theta_1-\theta_2)}{2-\cos\theta_1-\cos\theta_2+a.\cos(\theta_1-\theta_2)}$$

with $a = 1-\frac{1}{2}\sqrt{2}$ and for the ILU-7:

(3.2) $$\mu(\theta) = \frac{b.\cos(2\theta_1-\theta_2)}{2-\cos\theta_1-\cos\theta_2+b.\cos(2\theta_1-\theta_2)}$$

with $b = 0.11181$. The corresponding smoothing factors for ILU-5 and ILU-7 are $\bar{\mu} = 0.204$ and $\bar{\mu} = 0.126$ respectively. In the following table we assume that these smoothing factors are representative for the general cases: the

5-point and 7-point discretization of a general elliptic equation with variable coefficients of the same order of magnitude. The table gives smoothing factors, factors $\bar{\mu}$, numbers of operations per grid point per iteration step (n_i, i = 1,2,3) and numbers of operations per grid point for 10^{-1} reduction of the error ($n_i/|\log \bar{\mu}|$, i = 1,2,3).

Method	Poisson			General 5-point		General 7-point	
	$\bar{\mu}$	n_1	$n_1/\|\log \bar{\mu}\|$	n_2	$n_2/\|\log \bar{\mu}\|$	n_3	$n_3/\|\log \bar{\mu}\|$
Point Gauss-Seidel	0.50	5	16.6	9	29.9	13	43.2
Line Gauss-Seidel	0.447	8	22.9	14	40.1	18	51.5
ILU-5	0.204	11	15.9	14	20.3	14	20.3
ILU-7	0.126	15	16.7	18	20.0	18	20.0

Table 3.1. Smoothing factors and estimate of the number of operations per grid point for 10^{-1} reduction of the error.

On the basis of this table ILU-5 and ILU-7 are better than the two Gauss-Seidel methods for the general cases. In the case of singularly perturbed problems smoothing analysis demonstrates that incomplete LU-decomposition is less sensitive to ordering of grid points and other directional effects than Gauss-Seidel (see HEMKER (1980)).

The number of operations in one multigrid iteration with the adopted strategy in Chapter 2 is

Poisson: $27\frac{2}{3}$ operations/gridpoint

General 5- or 7-point case: $31\frac{2}{3}$ operations/gridpoint.

Results of numerical experiments with this multigrid method will be given. The multigrid iterations are terminated when the maximum of the difference between two iterands is smaller than 10^{-6}

$$|z_1^{(\sigma)}| = |(u^{\ell})^{(\sigma+1)} - (u^{\ell})^{(\sigma)}| < 10^{-6}. \tag{3.3}$$

Furthermore, we define the average reduction factor:

$$r_{av} = \left(\frac{|z_1^{(\sigma)}|}{|z_1^{(0)}|}\right)^{\frac{1}{\sigma}} \qquad \sigma \neq 0 \tag{3.4}$$

where σ is the smallest integer such that (3.3) holds.

Table 3.2 gives the average reduction factors for some elliptic problems The functions f and the boundary conditions are chosen so that the exact solution in column 2 is approximated. The problems are valid in the unit square. The mesh width of the finest grid is h = 1/32. All problems are discretized by means of central differences, except $\partial\omega/\partial x$ and $\partial\omega/\partial y$ in problem 4. They are discretized with upwind differences.

Equation	Exact solution	r_{av}
1. $\Delta\omega = f$	$\omega=\sin(x)\cdot e^y$	0.018
2. $\Delta\omega = 0$ $\Delta\psi = \omega$	$\omega=4\cos(x)\cdot\sinh(y)$ $\psi=2x\sin(x)\cdot\sinh(y)$	0.017
3. $\partial/\partial x\{(1+\sin x)\omega_x\}+\partial/\partial y\{(1+xy)\omega_y\}-\omega=f$	$\omega=y(x+\cos(x))$	0.015
4. $\partial\omega/\partial x - \partial\omega/\partial y = \Delta\omega/Re$; $Re = 10^{-4}$	$\omega=(1-e^{Rex}).(1-e^{-Rey})/(1-e^{Re})^2$	0.073

Table 3.2. r_{av} for some problems

Note that the r_{av} for the last, singular perturbed problem, is not much greater than for the other problems.

We can make an estimate of the number of operations for 10^{-1} reduction of the error for the Poisson equation:

$$27\tfrac{2}{3}\cdot 1/|\log 0.018| \cong 15.9 \text{ operation/gridpoint}$$

Compare with BRANDT (1977): ≈ 28 operations/gridpoint and NICOLAIDES (1979): 30-35 operations/gridpoint:

4. APPLICATION: THE NAVIER-STOKES EQUATIONS

Consider the driven square cavity flow with the Navier-Stokes equations in (ω,ψ)-formulation:

$$\left.\begin{aligned}\frac{\partial(\psi,\omega)}{\partial(y,x)} &= \frac{1}{Re}\Delta\omega \\ \Delta\psi &= \omega\end{aligned}\right\} \quad (x,y) \in \Omega, \tag{4.1}$$

with boundary conditions

$$\psi = 0, \quad \frac{\partial\psi}{\partial n} = g \tag{4.2}$$

with $g = 0$ on Γ_1, Γ_4, Γ_3 and $g = 1$ on Γ_2.

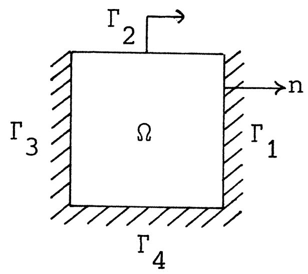


Fig. 4.1.

An equidistant computational frid Ω^ℓ is chosen:

$$\Omega^\ell = \{(x_i,y_j) \mid x_i = (i+1)(2^\ell+2)^{-1}, \quad y_j = (j+1)(2^\ell+2)^{-1}, \quad i = 0(1)2^\ell, \quad j = 0(1)2^\ell\} \tag{4.3}$$

There is a slight difference with (2.2) because the boundary conditions are substituted in the difference scheme. The equations (4.1) are discretized centrally except the first derivatives of ω, for instance $\partial\omega/\partial x$:

$$\left.\frac{\partial\omega}{\partial x}\right|_{i,j} = \frac{(1+\alpha_{ij})(\omega_{i+1,j}-\omega_{ij})+(1-\alpha_{ij})(\omega_{ij}-\omega_{i-1,j})}{2h} \tag{4.4}$$

with $h = (2\ell+2)^{-1}$ and α_{ij} the Il'in coefficient

$$\alpha_{ij} = -\coth\left(\frac{Re\left.\frac{\partial\psi}{\partial y}\right|_{ij}h}{2}\right) + \frac{2}{Re\left.\frac{\partial\psi}{\partial y}\right|_{ij}h}. \tag{4.5}$$

The boundary conditions for ω are found by combining $\Delta\psi = \omega$ and $\partial\psi/\partial n = g$:

$$(4.6) \qquad \omega_w = \frac{3}{h^2}\psi_{w+1} - \frac{1}{2}\omega_{w+1} + \frac{3}{h}g_w .$$

w is a point of $\partial\Omega$, w+1 indicates its nearest neighbour in Ω^ℓ in the direction of the normal.

The difference equations are Newton-linearized and the (linear) system in each Newton iteration is solved by the multigrid method. The termination criterium for the multigrid iterations is (3.3) and for the Newton iterations:

$$(4.7) \qquad |z_2^{(\rho)}| < |(u^\ell)^{(\rho+1)} - (u^\ell)^{(\rho)}| < 10^{-4}.$$

Experiments have been made for Reynolds numbers Re = 10, 50, 150. At Re = 10 we start with the zero solution, at the other Re-numbers with the solution of the preceding lower Re-numbers. The Table 4.1 gives $n^{(\rho)}$, the number of multigrid iterations and $r_{av}^{(\rho)}$, the average reduction factor in the ρ^{th} Newton iteration.

Note that r_{av} does not increase as $h \downarrow 0$ and is insensitive to changes in the coefficients induced by Newton iteration. Furthermore, r_{av} is comparable to r_{av} for the Poisson equation.

Another example is the flow around a cylinder with radius R and a uniform flow with velocity u_∞ at infinity. The non-dimensional Navier-Stokes equations are given in (4.8).

Re	h	$n^{(1)}$	$n^{(2)}$	$n^{(3)}$	$r_{av}^{(1)}$	$r_{av}^{(2)}$	$r_{av}^{(3)}$
10	1/6	4	2	-	0.027	0.026	-
	1/10	5	2	-	0.059	0.033	-
	1/18	5	2	-	0.056	0.034	-
	1/34	5	2	-	0.056	0.034	-
50	1/6	5	4	2	0.031	0.042	0.049
	1/10	5	4	1	0.052	0.061	0.056
	1/18	6	4	2	0.082	0.056	0.051
	1/34	6	4	2	0.083	0.062	0.052
150	1/6	7	6	3	0.099	0.092	0.079
	1/10	6	5	3	0.083	0.084	0.078
	1/18	6	5	2	0.080	0.083	0.064
	1/34	6	5	2	0.081	0.082	0.063

Table 4.1. Results for square cavity flow.

$$(4.8)\qquad \left.\begin{aligned} \frac{\partial(\psi,\omega)}{\partial(\eta,\xi)} &= \frac{1}{Re}\,\Delta_{\xi\eta}\omega \\ \Delta_{\xi\eta}\psi &= e^{2\xi}\omega \end{aligned}\right\}$$

with polar coordinates $x = e^{\xi}\cos\eta$, $y = e^{\xi}\sin\eta$. The Reynolds number is defined by:

$$(4.9)\qquad Re = \frac{u_\infty 2R}{\nu} .$$

The boundary conditions are:

$$(4.10)\qquad \left.\begin{aligned} &\xi = 0:\ \psi = \frac{\partial\psi}{\partial\xi} = 0 \\ &\eta = 0,\pi \text{ and } \xi \geq 0:\ \omega = \psi = 0 \\ &\xi = \pi:\ \omega = 0,\ \psi = e^{\xi}\sin\eta \end{aligned}\right\}$$

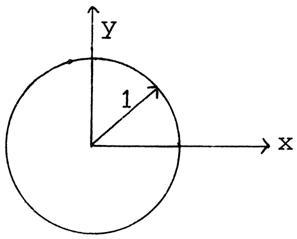


Fig. 4.2

The computational region is $\Omega = \{(\xi,\eta) \mid 0 \leq \xi \leq \pi,\ 0 \leq \eta \leq \pi\}$. The calculation is analogous with the calculation for the driven square cavity flow, so with Il'in upwind discretization. The results are presented in the following table:

Re	h	$n^{(1)}$	$n^{(2)}$	$n^{(3)}$	$r_{av}^{(1)}$	$r_{av}^{(2)}$	$r_{av}^{(3)}$
10	$\pi/6$	7	5	1	0.140	0.110	0.034
	$\pi/10$	7	5	1	0.139	0.120	0.053
	$\pi/18$	7	5	1	0.141	0.119	0.064
	$\pi/34$	8	5	1	0.170	0.100	0.029
50	$\pi/6$	8	5	1	0.178	0.106	0.042
	$\pi/10$	8	6	2	0.180	0.150	0.063
	$\pi/18$	8	6	3	0.175	0.153	0.068
	$\pi/34$	8	6	3	0.185	0.154	0.092
150	$\pi/6$	8	7	4	0.180	0.201	0.105
	$\pi/10$	8	6	4	0.185	0.160	0.105
	$\pi/18$	8	7	4	0.186	0.200	0.108
	$\pi/34$	8	7	4	0.187	0.195	0.110

Table 4.2. Results for the flow around a cylinder.

Although still fast, the average reduction factors are greater than in the previous cases, but they are still insensitive to h and to changes in the coefficients induced by Newton iteration.

5. CONCLUSION

A multigrid method has been presented that is fast and robust in the sense that it works for a large variety of elliptic problems without needing tuning or special modifications. The use of incomplete LU-decomposition makes it possible to treat uniformly elliptic and singularly perturbed problems by one and the same method. The combination of incomplete LU smoothing and Galerkin coarse grid approximation looks very promising.

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